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Measurement of Thermal Diffusivity by the Flash Method for a Two-Layer Composite Sample in the Case of Triangular Pulse[1]

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A method for measuring thermal diffusivity in one of the layers of a two-layer composite sample has been described. The heat transfer problem of a two-layer sample associated with pulse thermal diffusivity measurements has been analyzed for two cases: exponential and square-wave pulses. According to our measurements, a triangular heat-pulse function approximates reasonably well the output of the Nd–glass laser. In this paper, an expression is derived for the temperature transient at the rear face of two-layer sample being subjected to a triangular heat-pulse input on the front face. The analytical solution of the problem forms the basis of our method of data reduction. This solution has been programmed for computer processing of the data. The method described here has been successfully tested by limited measurements on copper and iron.

KEY WORDS: composites; copper; flash method; iron; thermal diffusivity.

1. INTRODUCTION

The flash method for measuring thermal diffusivity, originally proposed by Parker et al. [1] in 1961, has found widespread applications. Now the method has been extended to the measurement of thermal diffusivity in a two-layer composite sample. For example, the heat transfer problem of a two-layer sample has been analyzed for two cases: exponential and square-wave pulses. In 1968, Larson and Koyama [2] gave results for a particular exponential pulse characteristic of their flash tube. In 1974, Bulmer and Taylor [3] studied the problem for a square-wave pulse which they considered to be approximately of the form of a solid-state laser pulse. These

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two pulse functions are appropriate, however, only for some experimental situations. In fact, heat-pulse functions are different for different conditions. According to our measurements, a triangular heat-pulse function approximates reasonably well the output of the Nd–glass laser. Heckman [4] also believed that a triangular pulse will cover a broader range of experimental situations than the other two types of pulses. In this paper, an expression is derived for the temperature transient of a two-layer sample which is subjected to a triangular heat-pulse input on the front face. The method described here has been successfully tested by limited measurements on copper and iron.

2. MATHEMATICAL MODEL

The two-layer composite sample is a slab, with the first layer consisting of one material and the second of another. It is desirable to use a two-layer sample for some materials such as oxide film and coating, which are too friable or too thin to make a single-layer sample. An appropriate model may be visualized as a composite slab of infinite extension in the radial direction as shown in Fig. 1. The interface between the two media is located at $x = 0$. It is assumed that there is no contact resistance at the surface of separation. Furthermore, in the analysis presented, it is assumed that there is no radial flow of heat and no heat loss from either face, and all physical properties are independent of time and temperature. No finite pulse-time effects [5] are introduced, since no assumption is made regarding the instantaneous nature of the pulse of the heat-input source. Instead, a triangular heat-pulse function is assumed as shown in Fig. 2. It is often characteristic of this pulse that its peak intensity occurs at some time other than $t = 0$. For that reason, an adjustable parameter b, which locates

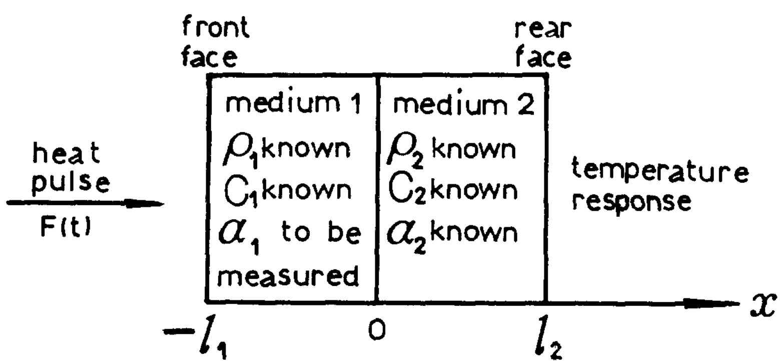

Fig. 1. Model of a two-layer sample.

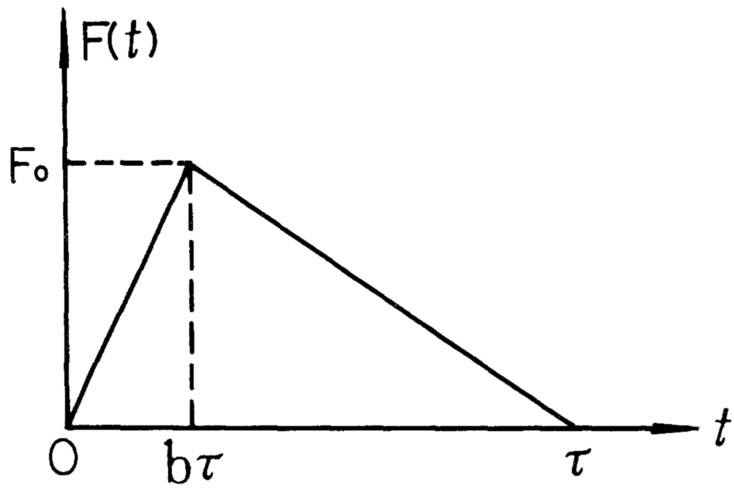

Fig. 2. The heat flux vs time for a triangular pulse.

the apex of the function, has been included. The analytic form of the pulse is

$$F(t)=\begin{cases}0 & (t\leqslant 0,\ t\geqslant \tau)\\ F_0 t/b\tau & (0\leqslant t\leqslant b\tau)\\ F_0(\tau-t)/(\tau-b\tau) & (b\tau\leqslant t\leqslant \tau)\end{cases} \tag{1}$$

This form is different from both the exponential pulse and the square-wave pulse.

With the above assumptions, heat flow in the sample is described by the equations

$$-\frac{\partial^2\theta_1}{\partial x^2}+\frac{1}{\alpha_1}\frac{\partial\theta_1}{\partial t}=0 \qquad (-1_1<x<0) \tag{2}$$

$$-\frac{\partial^2\theta_2}{\partial x^2}+\frac{1}{\alpha_2}\frac{\partial\theta_2}{\partial t}=0 \qquad (0<x<1_2) \tag{3}$$

where θ_j and α_j are the temperature excursion and thermal diffusivity, respectively, in the medium j ($j=1, 2$). The following initial and boundary conditions are assumed to apply:

$$\theta_1=0 \qquad (-1_1<x<0,\ t\leqslant 0) \tag{4}$$

$$\theta_2=0 \qquad (0<x<1_2,\ t\leqslant 0) \tag{5}$$

$$-\lambda_1\frac{\partial\theta_1}{\partial x}=F(t) \qquad (x=-1_1,\ t>0) \tag{6}$$

$$\frac{\partial \theta_2}{\partial x} = 0 \qquad (x = l_2, t > 0) \tag{7}$$

$$\theta_1 = \theta_2 \qquad (x = 0, t > 0) \tag{8}$$

$$\lambda_1 \frac{\partial \theta_1}{\partial x} = \lambda_2 \frac{\partial \theta_2}{\partial x} \qquad (x = 0, t > 0) \tag{9}$$

3. ANALYTICAL SOLUTION

The problem represented by Eqs. (2)–(9) has not been analytically treated in the literature. Unlike the exponential and square-wave pulses, the triangular pulse possesses a more complex Laplace transform, which makes the calculation very difficult. Since the transform of Eq. (1) was not available in the tables of Laplace transform, we rewrite Eq. (1) in another form, and finally, the Laplace transform of the pulse function is obtained as

$$\bar{F}(s) = \frac{F_0}{\tau s^2}\left[\frac{1}{b} - \frac{\exp(-b\tau s)}{b(1-b)} + \frac{\exp(-\tau s)}{1-b}\right] \tag{10}$$

Through the use of the Laplace transformation Eqs. (2)–(9) become

$$\frac{d^2\bar{\theta}_1}{dx^2} - \frac{s}{\alpha_1}\bar{\theta}_1 = 0 \qquad (-l_1 < x < 0) \tag{11}$$

$$\frac{d^2\bar{\theta}_2}{dx^2} - \frac{s}{\alpha_2}\bar{\theta}_2 = 0 \qquad (0 < x < l_2) \tag{12}$$

$$-\lambda_1 \frac{d\bar{\theta}_1}{dx} = \frac{F_0}{\tau s^2}\left[\frac{1}{b} - \frac{\exp(-b\tau s)}{b(1-b)} + \frac{\exp(-\tau s)}{1-b}\right] \qquad (x = -l_1, s > 0) \tag{13}$$

$$\frac{d\bar{\theta}_2}{dx} = 0 \qquad (x = l_2, s > 0) \tag{14}$$

$$\bar{\theta}_1 = \bar{\theta}_2 \qquad (x = 0, s > 0) \tag{15}$$

$$\lambda_1 \frac{d\bar{\theta}_1}{dx} = \lambda_2 \frac{d\bar{\theta}_2}{dx} \qquad (x = 0, s > 0) \tag{16}$$

where $\bar{\theta}_j(x, s) = L[\theta_j(x, t)]$ $(j = 1, 2)$. Solving these equations, one obtains the transformed solution in medium 2 as

$$\bar{\theta}_2(x, s) = \frac{\begin{pmatrix} F_0\{(1/b) - [\exp(-b\tau s)/b(1-b)] + [\exp(-\tau s)/(1-b)]\} \\ \times \operatorname{ch}[(U_2 s)^{1/2}(1 - x/l_2)] \end{pmatrix}}{\tau(\lambda_1 \rho_1 C_1)^{1/2} s^{5/2}[\operatorname{sh}(U_1 s)^{1/2} \operatorname{ch}(U_2 s)^{1/2} + \sigma \operatorname{ch}(U_1 s)^{1/2} \operatorname{sh}(U_2 s)^{1/2}]} \tag{17}$$

where

$$
\begin{aligned}
U_j &= \frac{l_j^2}{\alpha_j} \qquad (j=1,2) \\
\sigma &= \frac{X}{H} \\
X &= \left(\frac{U_1}{U_2}\right)^{1/2} \\
H &= \frac{C_1 l_1 \rho_1}{C_2 l_2 \rho_2}
\end{aligned}
\tag{18}
$$

Then the expression for the temperature excursion in medium 2 is given by the inversion integral as

$$\theta_2(x,t) = \frac{1}{2\pi i}\int_{X_0 - i\infty}^{X_0 + i\infty} \bar{\theta}_2(x,s)\, e^{st}\, ds \tag{19}$$

For convenience we take the temperature excursion in medium 2 relative to the final adiabatic equilibrium temperature difference θ_∞ and define a dimensionless variable $V_2(x, t)$ such that

$$V_2(x,t) = \frac{\theta_2(x,t)}{\theta_\infty} = \theta_2(x,t)\,\frac{2(\rho_1 c_1 l_1 + \rho_2 c_2 l_2)}{F_0 \tau} \tag{20}$$

Using Eq. (17) and Eq. (19), we obtain

$$V_2(x,t) = \frac{1}{2\pi i}\,\frac{2(U_1^{1/2} + \sigma U_2^{1/2})}{\tau^2}\int_{X_0 - i\infty}^{X_0 + i\infty} f(s)\, ds \tag{21}$$

where the integrand $f(s)$ is

$$f(s) = \frac{\left(\begin{array}{c}\{(1/b) - [\exp(-b\tau s)/b(1-b)] + [\exp(-\tau s)/(1-b)]\} \\ \times \operatorname{ch}[(U_2 s)^{1/2}(1 - x/l_2)]\exp(st)\end{array}\right)}{s^{5/2}[\operatorname{sh}(U_1 s)^{1/2}\operatorname{ch}(U_2 s)^{1/2} + \sigma\operatorname{ch}(U_1 s)^{1/2}\operatorname{sh}(U_2 s)^{1/2}]} \tag{22}$$

The integrand $f(s)$ has a pole of order three at $s = 0$. In addition, it has an infinite row of simple poles $s_k < 0$ along the negative real axis of the complex s plane. These s_k satisfy

$$\operatorname{sh}(U_1 s)^{1/2}\operatorname{ch}(U_2 s)^{1/2} + \sigma\operatorname{ch}(U_1 s)^{1/2}\operatorname{sh}(U_2 s)^{1/2} = 0 \tag{23}$$

Thus the vertical line $s = X_0 > 0$ is chosen as the path of integration of Eq. (21). To evaluate the integral of Eq. (21) the residue theorem is used. Thus

$$V_2(x, t) = \frac{2(U_1^{1/2} + \sigma U_2^{1/2})}{\tau^2}\left[\mathrm{Res}(0) + \sum_{k=1}^{\infty} \mathrm{Res}(s_k)\right] \tag{24}$$

where $\mathrm{Res}(s_k)$ is the residue of $f(s)$ at $s = s_k$. The residue at $s = 0$ may be evaluated as follows:

$$\mathrm{Res}(0) = \lim_{s \to 0} \frac{1}{2!}\frac{d^2}{ds^2}[s^3 f(s)] = \frac{\tau^2}{2}\frac{1}{U_1^{1/2} + \sigma U_2^{1/2}} \tag{25}$$

Evaluating this limit, we used the Maclaurin series of sh x and ch x many times. This procedure is very tedious. The residue of $f(s)$ at $s = s_k$ can be calculated as

$$\mathrm{Res}(s_k) = \lim_{s \to s_k} (s - s_k) f(s)$$

$$= \frac{\begin{pmatrix} 2\{(1/b) - [\exp(-b\tau s_k)/b(1-b)] + [\exp(-\tau s_k)/(1-b)]\} \\ \times \mathrm{ch}[(U_2 s_k)^{1/2}(1 - x/1_2)] \exp(s_k t) \end{pmatrix}}{s_k^2(U_1^{1/2} + \sigma U_2^{1/2})[\mathrm{ch}(U_1 s_k)^{1/2} \mathrm{ch}(U_2 s_k)^{1/2} + \Omega(x) \mathrm{sh}(U_1 s_k)^{1/2} \mathrm{sh}(U_2 s_k)^{1/2}]} \tag{26}$$

where

$$\Omega(x) = \frac{X + HX^{-1}}{H + 1} \tag{27}$$

For convenience we define the positive, real quantities β_k such that

$$(U_2 s_k)^{1/2} = i\beta_k \tag{28}$$

Therefore, Eq. (26) becomes

$$\mathrm{Res}(s_k) = \frac{\begin{pmatrix} 2U_2^2\{(1/b) - [\exp(b\tau\beta_k^2/U_2)/b(1-b)] + [\exp(\tau\beta_k^2/U_2)/(1-b)]\} \\ \cdot \exp(-\beta_k^2 t/U_2) \cdot \cos[\beta_k(1 - x/1_2)] \end{pmatrix}}{\beta_k^4(U_1^{1/2} + \sigma U_2^{1/2})[\cos(\beta_k X)\cos\beta_k - \Omega(X)\sin(\beta_k X)\sin\beta_k]} \tag{29}$$

And Eq. (23) becomes

$$\sin(\beta_k X)\cos\beta_k + \sigma\cos(\beta_k X)\sin\beta_k = 0 \tag{30}$$

Using Eq. (25) and Eq. (29) in Eq. (24), we obtain

$$V_2(x, t) = 1 + \frac{4U_2^2}{\tau^2} \cdot \sum_{k=1}^{\infty} \frac{\{(1/b) - [\exp(b\tau\beta_k^2/U_2)/b(1-b)] + [\exp(\tau\beta_k^2/U_2)/(1-b)]\}}{\beta_k^4[\cos(\beta_k X)\cos\beta_k - \Omega(X)\sin(\beta_k X)\sin\beta_k]} \cdot \exp(-\beta_k^2 t/U_2)\cos[\beta_k(1 - x/1_2)] \quad (31)$$

Thus, the expression for the transient at the rear surface of a two-layer sample for the case of triangular pulse may be obtained from Eq. (31) as $x = 1_2$.

$$V_2(t) = 1 + \frac{4U_2^2}{\tau^2} \cdot \sum_{k=1}^{\infty} \frac{\left(\begin{array}{c}\{(1/b) - [\exp(b\tau\beta_k^2/U_2)/b(1-b)] + [\exp(\tau\beta_k^2/U_2)/(1-b)]\} \\ \cdot \exp(-\beta_k^2 t/U_2)\end{array}\right)}{\beta_k^4[\cos(\beta_k X)\cos\beta_k - \Omega(X)\sin(\beta_k X)\sin\beta_k]} \quad (32)$$

From Eq. (32) the corresponding expression of a single-layer sample may be deduced. As an indirect check of the validity of the present analysis, we indicate that if the material properties in both media are set equal and the two layers are each taken as half the overall thickness, i.e., $1_1 = 1_2 = 1/2$, Eq. (32) yields the solution of the one-layer sample. With the above assumption from Eqs. (18), (27), (30), and (32) we obtain

$$V(t) = 1 + 2\sum_{k=1}^{\infty} (-1)^k \frac{2t_c^2}{k^4\pi^4\tau^2} \cdot \left[\frac{1}{b} - \frac{\exp(k^2\pi^2 b\tau/t_c)}{b(1-b)} + \frac{\exp(k^2\pi^2\tau/t_c)}{1-b}\right] \cdot \exp(-k^2\pi^2 t/t_c) \quad (33)$$

where $t_c = 1^2/\alpha$. Equation (33) is identical to the expression previously derived by Heckman [4] for the rear-face temperature excursion of the one-layer sample whose front face is heated by a triangular pulse.

4. EXPERIMENTAL METHOD

Equations (30) and (32) have been programmed for calculation on a computer. In the experiment measuring the value of $t_{1/2}$, which is the time

Table I. Experimental Results

Medium	Thickness (cm)	Density ($g \cdot cm^{-3}$)	Specific heat ($J \cdot kg^{-1} \cdot K^{-1}$)	Thermal diffusivity ($cm^2 \cdot s^{-1}$) Known	Thermal diffusivity ($cm^2 \cdot s^{-1}$) Calculated
Copper	0.403	8.9	385	1.12	1.09
Industrial pure iron	0.202	7.8	440	0.21	0.208

at which the rear-face temperature reaches half-maximum, together with the values of ρ_1, c_1, l_1, ρ_2, c_2, l_2, α_2, τ, and b are read in as data to the computer. Thus, unknown α_1 is obtained by an iterative method.

The laser thermal conductivity equipment employed uses a Nd–glass laser as the heat-pulse source. Its parameters corresponding to the triangular pulse are as follows: $b = 0.1$ and $\tau = 7.8 \times 10^{-4}$ s.

The composite samples of copper and industrial pure iron were fabricated. The thermal diffusivities of each component are known. The sheets of copper and iron were joined with tin solder whose thickness is about 0.006 cm. We regard the layer of the solder as iron in the course of calculation because its thickness is small and its diffusivity approximates that of iron. First, it was assumed that the diffusivity of copper was known and the diffusivity of iron was to be calculated; and second, the opposite was assumed.

The experimental results at room temperature are summarized in Table I. From the table it may be seen that the results obtained by the present method are in reasonably good agreement with the generally accepted values; the uncertainty is estimated to be about 3%. In conclusion, the present work indicates that the method has been successfully tested and that it may be considered to be acceptable.

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